Inequalities for generalized parton distributions H and E

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Abstract

Positivity bounds are derived for quark and gluon generalized parton distributions H and E which enhance the earlier inequalities.

Generalized parton distributions (GPDs) appear in the QCD description of a number of hard exclusive processes [1, 2, 3, 4, 5]. Since our current knowledge about GPDs is rather poor, any additional information deserves attention and from this point of view the so called positivity bounds [6, 7, 8] for GPDs are of certain interest.

In this paper the following bound for quark generalized parton distributions E_q and H_q is derived

$$\left|H_q(x,\xi,t) - \frac{\xi^2}{1-\xi^2}E_q(x,\xi,t)\right|^2 + \left|\frac{\sqrt{t_0-t}}{2M\sqrt{1-\xi^2}}E_q(x,\xi,t)\right|^2 \leq \frac{q(x_1)q(x_2)}{1-\xi^2}\,. \quad (1)$$

Here the notations of X. Ji are used for GPDs H_q, E_q and their arguments x, ξ, t (see e.g. [9]). The usual (forward) unpolarized quark distributions $q(x)$ are taken at values of x

$$x_1 = \frac{x+\xi}{1+\xi}, \quad x_2 = \frac{x-\xi}{1-\xi}\,. \quad (2)$$

Parameter

$$t_0 = -\frac{4\xi^2 M^2}{1-\xi^2} \quad (3)$$

corresponds to the maximal value of the squared momentum transfer t ($t \leq t_0 \leq 0$).

Obviously the bound (1) is stronger than the following inequality derived in ref. [10]

$$\left|H_q(x,\xi,t) - \frac{\xi^2}{1-\xi^2}E_q(x,\xi,t)\right| \leq \sqrt{\frac{q(x_1)q(x_2)}{1-\xi^2}}\,, \quad (4)$$

We also see from inequality (1) that

$$\frac{\sqrt{t_0 - t}}{2M} \left| E_q(x,\xi,t) \right| \leq \sqrt{q(x_1) q(x_2)} \,. \tag{5}$$

This inequality is by a factor of $\sqrt{1-\xi^2}$ stronger than a similar inequality presented in ref. [10].

As noticed in ref. [10] the earlier bound suggested in refs. [7, 8]

$$\left| H_q(x,\xi,t) \right| \leq \sqrt{\frac{q(x_1) q(x_2)}{1-\xi^2}} \tag{6}$$

is not justified since in its derivation the E_q contribution was overlooked.

The derivation of inequality (1) starts from the positivity condition [7, 8, 10]

$$\left\| \sum_{k=1}^{2} b_\lambda^k \frac{1}{\sqrt{(P_k n)}} \int \frac{d\tau}{2\pi} e^{i\tau x_k (P_k n)} \psi_\alpha (\tau n) \left| P_k, \lambda \right\rangle \right\|^2 \geq 0 \,. \tag{7}$$

The lhs contains a superposition of two states obtained by acting with quark fields ψ on a nucleon state $|P_k, \lambda\rangle$ with momentum P_k and polarization λ; n is a light-cone vector and coefficients b_λ^k are arbitrary. Expanding this squared sum and summing over "good" spin components α of quark fields we obtain

$$\begin{gathered} b_{\lambda'}^{1*} b_\lambda^1 q(x_1) + b_{\lambda'}^{2*} b_\lambda^2 q(x_2) \\ + \sqrt{1-\xi^2} \left[b_{\lambda'}^{1*} \mathcal{H}^q_{\lambda'\lambda}(-\Delta) b_\lambda^2 + b_{\lambda'}^{2*} \mathcal{H}^q_{\lambda'\lambda}(\Delta) b_\lambda^1 \right] \geq 0 \,. \end{gathered} \tag{8}$$

The summation over repeated indices is implied, $\Delta = P_2 - P_1$, and matrix $\mathcal{H}^q_{\lambda'\lambda}$ is taken from ref. [10]

$$\mathcal{H}^q_{\lambda'\lambda}(\Delta) = \begin{pmatrix} H_q - \frac{\xi^2}{1-\xi^2} E_q & \frac{-\Delta^1 + i\Delta^2}{2M(1-\xi^2)} E_q \\ \frac{\Delta^1 + i\Delta^2}{2M(1-\xi^2)} E_q & H_q - \frac{\xi^2}{1-\xi^2} E_q \end{pmatrix}_{\lambda'\lambda} \,. \tag{9}$$

Inequality (8) should hold for any b_λ^k. Taking special cases $b_-^1 = b_-^2 = 0$ or $b_-^1 = b_+^2 = 0$ one reproduces the bounds (4), (5).

Now let us consider the case of arbitrary b_λ^k. One can rewrite inequality (8) in the following form

$$\begin{gathered} \frac{1}{\sqrt{1-\xi^2}} \left[b_\lambda^{1*} b_\lambda^1 q(x_1) + b_\lambda^{2*} b_\lambda^2 q(x_2) \right] + \left(b_\lambda^{1*} b_\lambda^2 + b_\lambda^{2*} b_\lambda^1 \right) \left(H_q - \frac{\xi^2}{1-\xi^2} E_q \right) \\ + i \left(b_{\lambda_2}^{1*} b_{\lambda_1}^2 - b_{\lambda_2}^{2*} b_{\lambda_1}^1 \right) \left(\Delta^1 \sigma^2 - \Delta^2 \sigma^1 \right)_{\lambda_2 \lambda_1} \frac{E_q}{2M(1-\xi^2)} \geq 0 \end{gathered} \tag{10}$$

where σ^k are Pauli matrices. Rotating spinor indices λ of b_λ^k by the same $SU(2)$ transformation one can diagonalize $\Delta^1 \sigma^2 - \Delta^2 \sigma^1 \to |\Delta^\perp| \sigma^3$ so that the inequality takes the form

$$\frac{1}{\sqrt{1-\xi^2}} \left[b_\lambda^{1*} b_\lambda^1 q(x_1) + b_\lambda^{2*} b_\lambda^2 q(x_2) \right]$$

$$+\left(b_\lambda^{1*}b_\lambda^2+b_\lambda^{2*}b_\lambda^1\right)\left(H_q-\frac{\xi^2}{1-\xi^2}E_q\right)$$
$$+i\left(b_{\lambda_2}^{1*}b_{\lambda_1}^2-b_{\lambda_2}^{2*}b_{\lambda_1}^1\right)|\Delta^\perp|\sigma^3_{\lambda_2\lambda_1}\frac{E_q}{2M(1-\xi^2)}\geq 0\,. \tag{11}$$

We see that we have two independent inequalities for b_+^k and b_-^k

$$b_\pm^{k'}\left(\begin{array}{cc}\frac{q(x_1)}{\sqrt{1-\xi^2}} & H_q-\frac{\xi^2}{1-\xi^2}E_q\pm i\frac{|\Delta^\perp|E_q}{2M(1-\xi^2)}\\ H_q-\frac{\xi^2}{1-\xi^2}E_q\mp i\frac{|\Delta^\perp|E_q}{2M(1-\xi^2)} & \frac{q(x_2)}{\sqrt{1-\xi^2}}\end{array}\right)_{k'k}b_\pm^k\geq 0\,. \tag{12}$$

The positivity of this quadratic form immediately leads us to the constraint

$$\left(H_q-\frac{\xi^2}{1-\xi^2}E_q\right)^2+\left(|\Delta^\perp|\frac{E_q}{2M(1-\xi^2)}\right)^2\leq\frac{q(x_1)q(x_2)}{1-\xi^2}\,. \tag{13}$$

Taking into account that

$$t=\frac{4\xi^2M^2+|\Delta^\perp|^2}{1-\xi^2} \tag{14}$$

one arrives at the result (1).

Note that excluding E_q from the bound (1) we obtain a constraint on H_q

$$|H_q(x,\xi,t)|\leq\sqrt{\left(1-\frac{t_0}{t}\right)^{-1}\frac{q(x_1)q(x_2)}{1-\xi^2}}\,. \tag{15}$$

Since $t\leq t_0\leq 0$ this bound is weaker than the obsolete bound (6).

In a similar way one can derive the following bound for the gluon GPDs H_g, E_g (again using conventions of ref. [9])

$$\left|H_g(x,\xi,t)-\frac{\xi^2}{1-\xi^2}E_g(x,\xi,t)\right|^2$$
$$+\left|\frac{\sqrt{t_0-t}}{2M\sqrt{1-\xi^2}}E_g(x,\xi,t)\right|^2\leq\frac{x^2-\xi^2}{x^2(1-\xi^2)}g(x_1)g(x_2) \tag{16}$$

where $g(x)$ is the forward gluon distribution. In particular, this allows to reproduce the following result from ref. [10]

$$\left|H_g(x,\xi,t)-\frac{\xi^2}{1-\xi^2}E_g(x,\xi,t)\right|\leq\frac{1}{x}\sqrt{\frac{x^2-\xi^2}{1-\xi^2}g(x_1)g(x_2)}\,. \tag{17}$$

Keep in mind that the authors of ref. [10] use another normalization of H_g, E_g which differs by a factor of x.

By analogy with (15) we find from (16)

$$|H_g(x,\xi,t)|\leq\sqrt{\left(1-\frac{t_0}{t}\right)^{-1}\frac{x^2-\xi^2}{x^2}\frac{g(x_1)g(x_2)}{1-\xi^2}}\,. \tag{18}$$

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